Hybrid Multilevel/Multigrid Potential Preconditioner for Fast Finite Element Modeling

Yu Zhu, *Student Member, IEEE*, and Andreas C. Cangellaris, *Fellow, IEEE*

***Abstract*—A robust hybrid multilevel/multigrid potential preconditioner is introduced for the fast and robust finite-element modeling of electromagnetic structures. The proposed preconditioning process combines the advantages of the hierarchical multilevel preconditioner and the nested multigrid potential preconditioner into a novel preconditioner with superior computational versatility. Numerical experiments from the application of the new preconditioner to the finite-element analysis of microwave devices demonstrate its superior numerical convergence and efficient memory usage.**

***Index Terms*—Finite-element method, hierarchical multilevel, hybrid preconditioner, nested multigrid, vector and scalar potential formulation.**

I. INTRODUCTION

THE reasons for the slow convergence of the iterative solution of the finite-element approximation of the electrodynamic problem are by now well understood. They are associated with the dc modes contained in the null space of the curl operator and with the ill-conditioning of the finite-element method (FEM) matrix resulting from the oversampling of some of the low-frequency physical modes [1]. As it was proposed in [2], the spurious dc modes can be suppressed through the introduction of a spurious electric charge and the explicit imposition of the divergence-free condition, $\nabla \cdot \vec{D} = 0$, for the electromagnetic field. On the other hand, the difficulties associated with low-frequency physical modes can be addressed effectively by solving problems tentatively on coarser grids as in [3] and [4] or, equivalently, in lower order basis function spaces as in [6]. More specifically, those modes that are oversampled on the original FEM grid can be solved without loss of accuracy using FEM approximations with much fewer degrees of freedom.

The nested multigrid potential preconditioner uses a set of nested grids obtained by dividing each tetrahedron in the coarser grid into eight equal-volume subtetrahedra. Its application to electromagnetic problems appeared in [3] and [5]. However, the edge elements used in the nested multigrid technique are the lowest order basis functions [7]. Since they are not as effective in reducing the numerical dispersion error as higher order basis functions, the hierarchical multilevel potential preconditioner was proposed in [6] as an alternative. This latter technique uses one grid and a sets of hierarchical basis functions, i.e., $H^0(\mathrm{curl})$ and $H^1(\mathrm{curl})$, for enhanced accuracy. However, it requires the direct solution to the FEM matrix obtained from the approximation of the problem on the $H^0(\mathrm{curl})$ space. This limits its capability to solve electrically large structures.

To overcome these shortcomings of poor accuracy for multigrid preconditioner and factorization of large matrix for multilevel preconditioner, we propose a hybrid preconditioner, which uses hierarchical multilevel technique on the top of nested multigrid one so that it increases the the accuracy of multigrid FEM on one hand and decreases the factorization cost of multilevel FEM by further shrinking the size of matrix for LU on the other hand.

Manuscript received December 5, 2001; revised May 21, 2002. The review of this letter was arranged by Associate Editor Dr. Ruediger Vahldieck.

The authors are with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: cangella@uiuc.ed).

Publisher Item Identifier 10.1109/LMWC.2002.801943.

II. HYBRID MULTILEVEL/MULTIGRID POTENTIAL PRECONDITIONER

Consider the solution of the following vector wave equation of a three-dimensional electromagnetic device:

$$\nabla \times \nabla \times \vec{E} - \omega^2 \epsilon \mu \vec{E} = 0. \tag{1}$$

The following FEM matrix equation is obtained by a Galerkin's procedure [6]:

$$M_{EE} x_E = f_E \tag{2}$$

where x_E contains the expansion coefficients for $\vec{E}$ and f_E is from the excitation on the driven port. As elaborated in [1] and [4], the field formulation has the deficiency of slow convergence when an iterative solver applies. One of the primary reasons for this is the presence of spurious dc modes in the null space of the curl operator. These spurious dc modes can be eliminated or suppressed by using the vector-scalar potential formulation to impose explicitly the divergence-free constraint on the field. Following [2], $\vec{E} = \vec{A} + \nabla V$, we obtain the matrix equation for potential formulation from (1) and divergence-free condition $\nabla \vec{D} = 0$

$$\begin{pmatrix} M_{AA} & M_{AV} \\ M_{VA} & M_{VV} \end{pmatrix} \begin{pmatrix} x_A \\ x_V \end{pmatrix} = \begin{pmatrix} f_A \\ f_V \end{pmatrix} \tag{3}$$

where x_A and x_V contain, respectively, the expansion coefficients for $\vec{A}$ and V. The field and potential formulations are equivalent. This equivalence allows the transformation from the matrix equation of field formulation (2) to the one of potential formulation (3), as discussed in [2] and [6]. During the iterative solution of (2), in order to render the search vectors divergence-free, the pseudoresidual equation $M_{EE} z_E = r_E$ has to be solved approximately using potential formulation by the following procedures. First, the matrix (2) of field formulation is transformed to the one of the potential formulation (3). Next, in the potential formulation, the matrix (3) is solved in two steps.

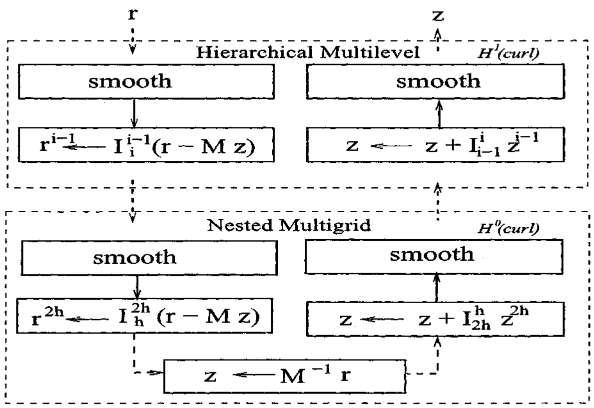

Fig. 1. Flowchart of the proposed hybrid preconditioner.

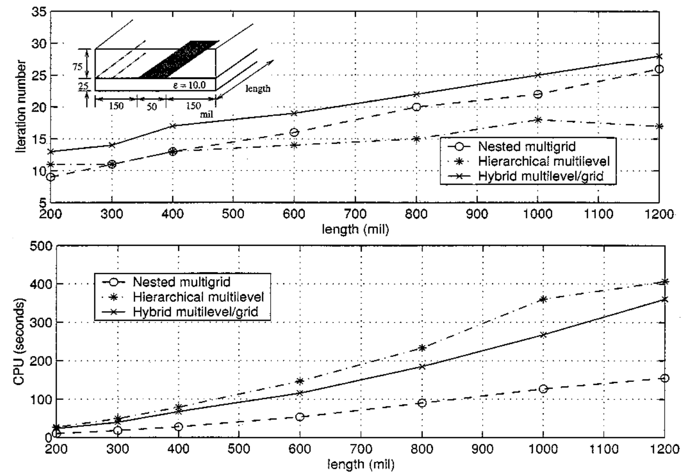

Fig. 2. Number of iterations and CPU time versus the length of the microstrip line.

Step 1 involves the solution of z_A in $M_{AA}z_A = r_A - M_{AV}z_V$; step 2 imposes explicitly the divergence-free field constraint in order to eliminate/suppress the spurious dc modes by solving z_V in $M_{VV}z_V = r_V - M_{VA}z_A$. Both are approximate solutions and implemented through the Gauss–Seidel method. Finally, after the solution in the potential formulation is obtained, it is transformed back to solution of the field formulation as discussed in [2] and [6].

Only to render the search vectors divergence-free is not enough for fast convergence of the iterative solution, since it cannot handle the low-frequency physical modes effectively [1]. It has to be combined with multigrid/multilevel techniques. The pseudocode, MG(z_E, r_E, i, j) and flowchart of the proposed hybrid potential preconditioner are in the following, where i and j is level number for multilevel and multigrid processes, respectively:

1. $z_E \leftarrow 0$,
2. if $i == 0$, // Nested Multigrid
 2a. if $j == 0$,
  then solve $M_{EE}^{h} z_E = r_E$ // coarsest grid
 2b. else
  2b.1 smooth(z_E,r_E) for v_1 times.
  2b.2 $r_E^{2h} \leftarrow I_h^{2h}(r_E - M_{EE}^{h} z_E)$ and $z_E \leftarrow 0$
  2b.3 MG $(z_E^{2h}, r_E^{2h}, 0, j-1)$
  2b.4 $z_E \leftarrow z_E + I_{2h}^{h} z_E^{2h}$
  2b.5 smooth(z_E, r_E) for v_2 times.
3. else // Hierarchical Multilevel
 3a. Smooth(z_E, r_E) for v_1 times.
 3b. $r_E^{i-1} \leftarrow I_i^{i-1}(r_E - M z_E)$ and $z_E^{i-1} \leftarrow 0$
 3c. MG $(z_E^{i-1}, r_E^{i-1}, i-1, j)$
 3d. $z_E \leftarrow z_E + I_{i-1}^{i} z_E^{i-1}$
 3e. Smooth(z_E, r_E) for v_2 times.

From the pseudocode and flowchart in Fig. 1, it is obvious the nested multigrid preconditioning solution is placed in the solution of the $H^0(\mathrm{curl})$ matrix equation of the hierarchical multilevel preconditioner so that the matrix for the factorization is further shrunk. The smoothing operations in both the nested and the hierarchical preconditioning processes are performed on the matrix equation of the potential formation as of discussed before.

I_i^{i-1} and I_{i-1}^{i} denote the interlevel operators that map the residual and the correction between two adjacent levels. Since the hierarchical basis functions are used, the construction of the two operators is trivial. I_h^{2h} and I_{2h}^{h} are the intergrid operators. They are the transformation matrix of the two sets of the basis function spaces between two adjacent nested grids and its transpose, respectively [4].

III. Numerical Results

The proposed hybrid preconditioner can be combined with a Krylov subspace-based iterative solver. The stopping criterion used is $\|r\|_2/\|b\|_2 = tol$, where $tol = 1.0e{-}6.0$ and b is the right-hand side vector of the matrix equation assuming that the initial guess vector is zero. The number of presmoothing and post-smoothing operations is taken to be $v1 = v2 = 3$. The calculations are done on a Pentium III 600 MHz PC.

Fig. 2 depicts the number of iterations and the required CPU time versus the length of an unbounded microstrip line. The operating frequency is 20 GHz. The length of the line is increased from 200 mils to 1200 mils ($\sim 10.0\lambda$). This results in an increase in the number of unknowns from 24 936 to 160 962. The average spatial resolution in the coarsest grid is ~ 3.5 pts/λ.

The hybrid preconditioner uses a two-level hierarchical and a two-grid nested process. The hierarchical multilevel preconditioner uses two sets of basis functions, $H^1(\mathrm{curl})$ and $H^0(\mathrm{curl})$. Contrast to the hybrid one, the direct solution for the hierarchical preconditioner is called once after the pseudoresidual equation is mapped to $H^0(\mathrm{curl})$; while for the hybrid preconditioner, the equation is further shrunk to the coarser grid. Therefore, the hierarchical preconditioner corresponds to the upper block in the flowchart of Fig. 1. The third preconditioner is the nested multigrid technique which is to solve the problem only using $H^0(\mathrm{curl})$ the preconditioner uses two sets of nested multigrids. It corresponds to the lower block in the flowchart of Fig. 1.

From the comparison of the three preconditioners in Fig. 2, it is clear that both the number of iterations and the CPU time

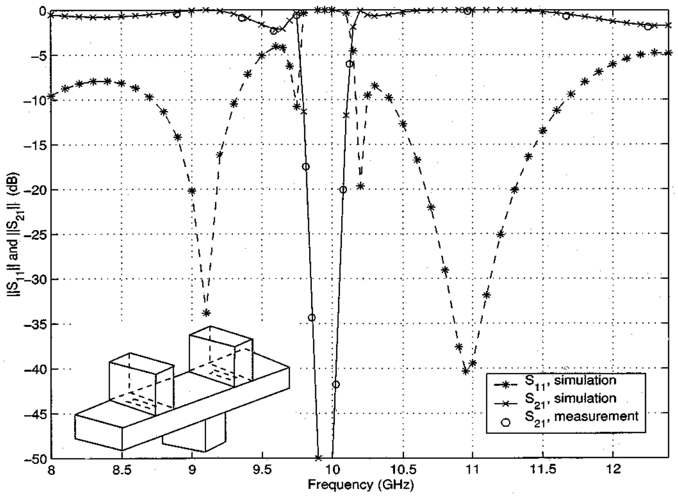

Fig. 3. Scattering parameters of the bandstop filter. Waveguide dimensions: 22.86×10.16 mm^2. Length between resonators $=$ 19.63 mm. Height of the outer resonators $=$ 16.54 mm; of the inner resonator $=$ 16.94 mm. Iris thickness $=$ 0.0 mm. Outer iris width $=$ 12.22 mm. Inner iris width $=$ 11.63 mm. Height of all irises $=$ 3.05 mm.

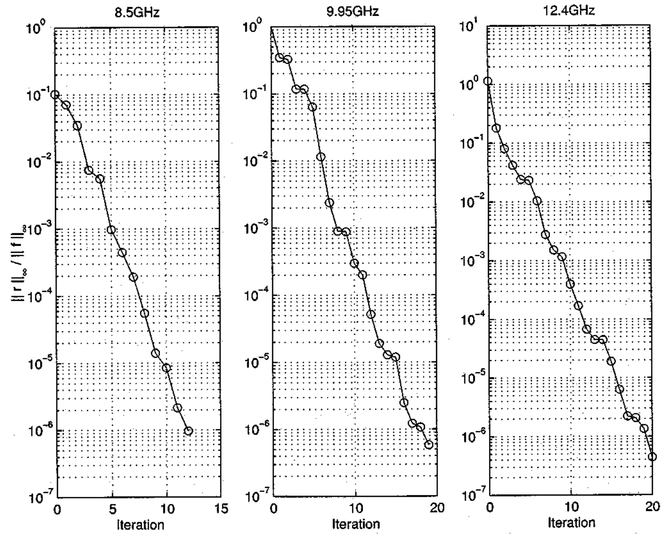

Fig. 4. Number of iterations at various frequencies.

increase with the line length, due to the numerical dispersion error which is known to increase with electrical length. The hierarchical multilevel one exhibits the best convergence performance. Obviously, it performs better than the hybrid one, because it solves the $H^0(\mathrm{curl})$ matrix directly rather than solving it approximately by mapping it down to a coarser grid as done by the hybrid preconditioner. However, by doing so it factorizes a larger matrix. For example, for the 1200-mil microstrip line, there are 160 962 and 30 209 unknowns on the $H^1(\mathrm{curl})$ and $H^0(\mathrm{curl})$ level, respectively. For the hierarchical multilevel preconditioner, it factorizes the $H^0(\mathrm{curl})$ matrix of the size 30 209; while for the hybrid preconditioner, it further maps the $H^0(\mathrm{curl})$ matrix down to the one on the coarser grid, so that the size of matrix for factorization is shrunk by about one eighth, e.g., 3930. Because of the further mapping onto coarser grid, the hybrid preconditioner shows slower convergence than the hierarchical multilevel one. However, the gain is a much smaller matrix to be factorized. Thus, the over CPU time is shortened and memory requirement is less. For examples, for the 1200-mil line, the total memory requirement is 60 MB for the hybrid preconditioner and 112 MB for the hierarchical multilevel one.

The next example considers the bandstop waveguide filter of [8] (see Fig. 3). The numbers of unknowns on $H^1(\mathrm{curl})$ and $H^0(\mathrm{curl})$ are 152 062 and 26 375. The average coarsest grid resolution is $\sim$ 5.1 pts$/\lambda$ at 12.4 GHz. Again, the hybrid preconditioner uses a two-level and two-grid scheme. Thus the size of matrix for factorization is 2784. The calculated scattering parameters are in excellent agreement with measured results. The convergence behavior at various frequencies is shown in Fig. 4. On the average, CPU time is about 150 s, while the total memory requirement is 53 MB.

IV. Concluding Remarks

An efficient hybrid multilevel/multigrid potential preconditioner has been introduced and demonstrated for the robust and expedient finite–element analysis of electromagnetic devices. Through the combination of hierarchical multilevel and nested multigrid techniques, the size of the matrix that requires LU factorization is dramatically reduced, while the accuracy of the FEM solution is guaranteed by using high-order spaces for the expansion of the fields. Combined with *hp*-adaptive mesh refinement, the proposed hybrid preconditioner will enable robust, fast, and accurate finite–element modeling of electrically large electromagnetic structures with high geometric complexity.

References

[1] R. Dyczij-Edlinger and O. Biro, "A joint vector and scalar potential formulation for driven high frequency problems using hybrid edge and nodal finite elements," *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 15–23, Jan. 1996.

[2] R. Dyczij-Edlinger, G. Peng, and J. F. Lee, "A fast vector-potential method using tangentially continuous vector finite elements," *IEEE Trans. Microwave Theory Tech.*, vol. 46, pp. 863–868, June 1999.

[3] R. Beck and R. Hiptmair, "Multilevel solution of the time-harmonic Maxwell's equations based on edge elements," *Int. J. Numer. Meth. Eng.*, vol. 45, pp. 901–920, 1999.

[4] Y. Zhu and A. C. Cangellaris, "Nested multigrid vector and scalar potential finite element method for fast computation of two-dimensional electromagnetic scattering," *IEEE Trans. Antennas Propagat.*, Oct. 2002, to be published.

[5] ——, "Robust multigrid preconditioner for fast finite element modeling of microwave devices," *IEEE Microwave Wireless Compon. Lett.*, vol. 11, pp. 416–418, Oct. 2001.

[6] Y. Zhu and A. Cangellaris, "Hierarchical multilevel potential preconditioner for fast finite element modeling of microwave devices," *IEEE Trans. Microwave Theory Tech.*, vol. 50, pp. 1984–1989, Aug. 2002.

[7] Y. Zhu and A. C. Cangellaris, "Hierarchical finite element basis function spaces for tetrahedral elements," in *Appl. Comput. Electromagn. Soc. Meeting*, Monterey, CA, Mar. 2001, pp. 69–74.

[8] T. Sieverding and F. Arndt, "Field theoretic CAD of open or aperture matched T-junction coupled rectangular waveguide structures," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 353–363, Feb. 1992.